

ON THE \mathbf{c} -VECTORS OF AN ACYCLIC CLUSTER ALGEBRA

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ABSTRACT. We prove that the set of \mathbf{c} -vectors of the cluster algebra associated to an acyclic quiver Q coincides with the set of real Schur roots and their opposites in the root system associated to Q .

1. INTRODUCTION

In the theory of cluster algebras, a prominent role is played by two families of integer vectors, namely the \mathbf{c} - and the \mathbf{g} -vectors. They were first introduced in [8] in order to parametrize (respectively) the coefficients and the cluster variables of a (geometric) cluster algebra. In [24] the authors showed that both families were closely related provided that the \mathbf{c} -vectors satisfy the *sign-coherence* property, *i.e.* each \mathbf{c} -vector has either all its entries nonnegative or all its entries nonpositive. Moreover, many important conjectures about cluster algebras can be proved to be true if this last condition holds. The sign-coherence of the \mathbf{c} -vectors was proved in [7] for the case of skew-symmetric exchange matrices, using decorated representations of quivers with potentials (see [6]). Alternative proofs were given in [26] and [21]. For acyclic quivers, the clusters of \mathbf{c} -vectors were characterized in [27]. We know that \mathbf{c} -vectors are always dimension vectors of indecomposable rigid modules (over an appropriate algebra), see section 8 of [21], *cf.* Theorem 14 below. In the present note, we show the other inclusion, *i.e.* the set of positive \mathbf{c} -vectors associated to an acyclic quiver Q coincides with the set of real Schur roots in the root system associated to Q . This result can also be obtained [28] using the approach presented in [27]. A description of the \mathbf{c} -vectors for general quivers seems to be unknown. A non acyclic example is computed in [22].

Section 2 is devoted to recall the reader the constructions used along remind section 3 to prove our main result. In section 4 we present some interesting examples concerning possible generalizations of our main result.

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2. REMINDERS

Unless otherwise stated Q is an arbitrary finite quiver with n vertices and k is an algebraically closed field.

2.1. Dimension vectors and root systems. Denote by kQ the *path algebra* associated to Q and denote by $\text{mod}(kQ)$ the category of finite dimensional right kQ -modules (see [1] for background material). The category $\text{mod}(kQ)$ is equivalent to the category of finite-dimensional representations of Q^{op} over k . We write $\underline{\dim} M$ for the dimension vector of the representation corresponding to a module $M \in \text{mod}(kQ)$.

Dimension vectors are closely related with generalized root systems (which are associated to arbitrary quivers, see [14] for details).

Theorem 1. ([14]) *For any integer vector \underline{v} , there is an indecomposable kQ -module M with $\underline{\dim}(M) = \underline{v}$ if and only if \underline{v} is a positive root in the root system associated to Q .*

Definition 2. A *real Schur root* associated to Q is the dimension vector of an indecomposable kQ -module without self-extensions (*rigid*). These vectors are independent of k (see [5, Theorem 1]). We denote by $\Phi^{re, Sch}$ the set of real Schur roots.

2.2. Tilting theory and the cluster category. In this section, we assume that Q is acyclic. Denote by \mathcal{A} the category $\text{mod}(kQ)$.

Definition 3. We say that a module $T \in \mathcal{A}$ is *tilting* if and only if T is *rigid*, i.e. $\text{Ext}_{\mathcal{A}}^1(T, T) = 0$, and T is the direct sum of n pairwise non isomorphic indecomposable modules. For a tilting module T , we denote by B the algebra $\text{End}_{\mathcal{A}}(T)$ and by \mathcal{B} the category $\text{mod}(B)$.

Let C be a category of the form \mathcal{A} or \mathcal{B} . Then the Grothendieck group $K_0(C)$ admits an unique bilinear form $\langle \cdot, \cdot \rangle$ for which the basis of the simples is dual to the basis of the indecomposable projectives $\{P_1, \dots, P_n\}$ in the sense that $\langle [P_i], [S_j] \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. We call an isomorphism of Grothendieck groups an *isometry* if it respects the bilinear form. We denote by $\mathcal{D}^b(C)$ the bounded derived category of C and by Σ its canonical shift (or suspension) functor. The natural inclusion from C to its derived category induces an isomorphism between $K_0(C)$ and $K_0(\mathcal{D}^b(C))$, the Grothendieck group of $K_0(\mathcal{D}^b(C))$ as a *triangulated* category (cf. [13]). The following theorem will be very useful in the sequel.

Theorem 4. ([12, Section 2]) *Let T be a tilting module. Then the functor*

$$(2.1) \quad - \otimes_B^L T : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$$

is a triangle equivalence which induces an isometry of Grothendieck groups $K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A})$.

Definition 5. ([3]) Let \mathcal{C} be the *cluster category* associated to Q , that is, the orbit category $\mathcal{D}^b(\mathcal{A})/\tau^{-1} \circ \Sigma$, where τ denotes the AR-translation of $\mathcal{D}^b(\mathcal{A})$. The category \mathcal{C} admits a canonical triangulated structure [16] whose suspension functor is induced by Σ . An object X in \mathcal{C} is called *cluster tilting* if $\text{Hom}_{\mathcal{C}}(X, \Sigma X) = 0$ and X is the sum of n pairwise non isomorphic indecomposable objects.

Remark 6. By construction, there is a natural functor from $\text{mod}(kQ)$ to \mathcal{C} which is faithful but not full in general. By [3] we know that the image of every tilting module is a cluster tilting module.

2.3. Derived categories for dg-algebras. We follow the approach of [15]. Throughout this subsection, A is a dg-algebra over k with differential d_A . That is a \mathbb{Z} -graded k -algebra or equipped with a degree 1 k -linear differential. A dg-module M over A is a graded A -module endowed with a differential d_M satisfying

$$d_M(ma) = d_M(m)a + (-1)^{|m|}md_A(a)$$

for every homogeneous element m in M of degree $|m|$, and every a in A . Given two dg A -modules M and N , the *morphism complex* is the graded vector space $\text{Hom}_A(M, N)$ whose i -th component is the subspace of the product $\prod_{j \in \mathbb{Z}} \text{Hom}_k(M^j, N^{j+i})$ consisting of the morphisms f such that $f(ma) = f(m)a$, for all m in M and all a in A , and whose differential d is given by

$$d(f) = f \circ d_M + (-1)^{|f|}d_N \circ f$$

for a homogeneous morphism f of degree $|f|$.

Let $\mathcal{H}(A)$ be the *homotopy category* of dg A -modules. It has as objects the dg A -modules and as morphisms the 0-th homology groups of the morphism complexes, *i.e.* $\mathcal{H}(A)(M, N) = H^0(\mathcal{H}om_A(M, N))$. The *derived category* $\mathcal{D}(A)$ is the localization [9] of $\mathcal{H}A$ with respect to the full subcategory of acyclic dg A -modules. In other words, we are adding formal inverses to those morphisms of complexes inducing isomorphisms in all homology groups.

Remark 7. $\mathcal{D}(A)$ is an additive category. Moreover it is always triangulated [13], with *suspension functor* Σ given by the degree shift on complexes.

Remark 8. The derived category $\mathcal{D}(\text{Mod } A)$ of the abelian category $\text{Mod } A$ of all A -modules, where A is a k -algebra, is a classical object (*cf.* [11]). However, it can be obtained from this more general setting by thinking of A as a dg algebra concentrated in degree 0.

2.4. Quivers with potentials and the Ginzburg dg algebra. For this section, we assume Q has no loops nor two-cycles. Let \widehat{kQ} be the *completed path algebra* associated to Q , *i.e.* the completion of kQ with respect to the *path length*. As a topological algebra, \widehat{kQ} is endowed with the \mathfrak{m} -adic topology, where \mathfrak{m} is the ideal generated by the arrows in Q . A *potential* in Q is a (possibly infinite) linear combination of cyclic paths in Q , considered up to *cyclic equivalence* and whose summands are pairwise cyclically inequivalent (*cf.* [6, Section 2]). Potentials can be thought of as elements of the space

$$\text{Pot}(Q) = \widehat{kQ}/C,$$

where C is the closure in \widehat{kQ} of the subspace $[\widehat{kQ}, \widehat{kQ}]$. The only nonzero potentials considered in this note are the *reduced* ones, *i.e.* those which only involve cycles of length strictly greater than 2. For each arrow α of Q , we define the *cyclic derivative*

$$\partial_\alpha : \text{Pot}(Q) \longrightarrow \widehat{kQ}$$

as the continuous linear map taking the equivalence class of a cycle c to the sum

$$\sum_{c=u\alpha v} vu$$

taken over all possible cyclic paths $u\alpha v$ equal to c .

Definition 9. The *Jacobian algebra* $J(Q, W)$ associated to a *quiver with potential* (Q, W) (QP for short) is the quotient

$$J(Q, W) = \widehat{kQ}/\partial(W).$$

where $\partial(W)$ denotes the closure of the ideal generated by the elements $\partial_\alpha(W)$ as α ranges through all the arrows of Q .

The mutation operation on quivers admits an extension to the class of reduced quivers with potential up to an appropriate equivalence, namely the *right equivalence* (*cf.* Sections 4 and 5 of [6]). In contrast to quiver mutation, iterated QP mutation can produce 2-cycles in the quivers. We call a potential *non-degenerate* if this is not the case.

Now fix a quiver with potential (Q, W) . We recall the construction of the (completed) *Ginzburg dg algebra* $\Gamma_{Q, W} = \Gamma$ (for short) associated to (Q, W) (recall that a *dg algebra* over k is a \mathbb{Z} -graded k -algebra or equipped with a degree 1 k -linear differential). We let \overline{Q} be the graded quiver obtained from Q as follows:

- \overline{Q} has the same vertices as Q ;
- the set of arrows of degree 0 in \overline{Q} is the set of arrows of Q ;

- the set of arrows of degree -1 in \overline{Q} is formed by arrows $\alpha^* : j \rightarrow i$, for each arrow $\alpha : i \rightarrow j$ in Q ;
- the set of arrows of degree -2 in \overline{Q} is formed by loops $t_i : i \rightarrow i$, one for each vertex i of \overline{Q} ;
- these are the only arrows.

The underlying graded algebra of Γ is the graded completion of the graded path algebra $k\overline{Q}$ with respect to the *path degree* (lazy paths introduced below are considered as of degree 0). The differential d of Γ is the unique continuous linear differential acting as follows on the arrows:

- $d(\alpha) = 0$ for the arrows α of degree 0;
- $d(\alpha^*) = \partial_\alpha(W)$ for the arrows α^* of degree -1 ;
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each loop t_i of degree -2 . Here the sum is taken over all arrows of degree 0 and e_i is the (*lazy*) path of length zero that stays at vertex i .

This definition is slightly different from the one first introduced in [10] see [19, Remark 2.7].

Remark 10. The zeroth homology $H^0(\Gamma_{Q,W})$ is canonically isomorphic to $J(Q, W)$.

We refer to [19] for the effect of the mutation of QPs on the associated Ginzburg algebras.

3. MAIN RESULT

Theorem 11. *If Q is acyclic, then the set of \mathbf{c} -vectors associated to Q is equal to $\Phi^{re, Sch} \cup -\Phi^{re, Sch}$.*

We deduce the theorem from the following propositions.

Proposition 12. *If Q is acyclic and $T = \bigoplus T_i$ is a tilting module in $\text{mod } kQ$ and B the endomorphism algebra $\text{End}_{kQ}(T)$, then the image of each simple B -module under the isometry induced by the equivalence $F = - \otimes_B^L T : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$ (see Theorem 4) is a \mathbf{c} -vector.*

Proof. Let \mathcal{C}_Q be the cluster category associated to Q . Let $\tilde{T} = \bigoplus \tilde{T}_i$ be the image of T in \mathcal{C}_Q . Thus, \tilde{T} is a cluster tilting object. Denote by $(g_{ij}^{\tilde{T}})$ (resp. $(c_{ij}^{\tilde{T}})$, with $1 \leq i, j \leq n$), the \mathbf{g} -matrix (resp. \mathbf{c} -matrix) associated to \tilde{T} . It is easy to see that under the equivalence F , the class of B is mapped onto the class of T , and the classes of the indecomposable projective modules P_i^B are mapped onto the indecomposable factors T_i of T . By [17, Corollary 6.8], we know that $[T_i] = \Sigma g_{ji}^{\tilde{T}} [P_j^A]$. This implies that $F[S_i^B] = \Sigma c_{ji}^{\tilde{T}} [S_j^A]$. Indeed, let $F[S_i^B] = \Sigma n_{ji} [S_j^A]$. Then

$$\begin{aligned} \delta_{ij} &= \langle F[P_i^B], F[S_j^B] \rangle \\ &= \langle \Sigma g_{ki}^{\tilde{T}} [P_k^A], \Sigma n_{kj} [S_k^A] \rangle \\ &= \Sigma g_{ki}^{\tilde{T}} n_{kj}, \end{aligned}$$

and by [24, Theorem 1.2], we obtain $n_{ij} = c_{ij}^{\tilde{T}}$ for $1 \leq i, j \leq n$. □

Proposition 13. *Each root in $\Phi^{re, Sch} \cup -\Phi^{re, Sch}$ is a \mathbf{c} -vector.*

Proof. Let M be a non injective rigid indecomposable kQ -module and let $H = kQ$. We consider the (dual version of the) Bongartz exact sequence associated to M , i.e. the universal extension

$$0 \rightarrow M^r \rightarrow G \rightarrow DH \rightarrow 0$$

of DH by an object of $\text{add}(M)$ (see [2] or [1, VI.2.]). We know that $T := G \oplus M$ is a tilting module. Moreover, G is a projective generator of the abelian category $M^\perp = \{N : \text{Hom}(M, N) = 0 = \text{Ext}^1(M, N)\}$. In particular, the vertex corresponding to M of the quiver of $B = \text{End}_k Q(T)$ is a source. If we denote by S_M the simple projective B -module associated to this vertex, then $F[S_M] = M$, and therefore, $\underline{\dim}(M)$ is a \mathbf{c} -vector by Proposition 12. For an injective indecomposable I , we chose a tilting complex T by completing I into a section with source I of the AR-quiver of $\mathcal{D}^b(A)$. For $B = \text{End}(T)$, we have a triangle equivalence $F : \mathcal{D}^b(B) \rightarrow \mathcal{D}^b(A)$ taking a simple projective to I . As above, we conclude that $\underline{\dim} I$ is a \mathbf{c} -vector. \square

The remaining inclusion is immediate from the following general fact.

Theorem 14. ([21]) *Let Q be a quiver without loops nor 2-cycles and W a non degenerate potential on Q . Then each positive \mathbf{c} -vector of Q is the dimension vector of a finite dimensional rigid module with endomorphism algebra k over the Jacobian algebra of (Q, W) .*

Proof. We follow the approach of [21] as presented in section 7.7 of [18]. We denote by $J(Q, W)$ the Jacobian algebra associated to the opposite quiver (Q^{op}, W^{op}) and by Γ the corresponding Ginzburg dg algebra. We consider $\mathcal{D}_{fd}(\Gamma)$ the full sub-category of $\mathcal{D}(\Gamma)$ formed by the dg modules whose homology is of finite total dimension. The homology induces a natural t -structure in $\mathcal{D}_{fd}(\Gamma)$, which admits $\mathcal{A} = \text{mod } J(Q, W)$ as heart. Let \mathbb{T}_n denote the n -regular tree. If we assign to a fixed vertex $t_0 \in \mathbb{T}_n$ the quiver with potential (Q, W) , then, by iterated mutation, we can associate a quiver with potential $(Q(t), W(t))$ and a \mathbf{c} -matrix $C(t)$ to any vertex $t \in \mathbb{T}_n$. We write $\Gamma(t)$ for the Ginzburg dg algebra associated to the opposite of $(Q(t), W(t))$. We denote by

$$\Phi(t) : \mathcal{D}(\Gamma(t)) \rightarrow \mathcal{D}(\Gamma)$$

the triangle equivalence constructed in section 7.7 of [18]. By parts *a*) and *b*) of Theorem 7.9 of [18], *cf.* also Remark 8.2 of [21], we have

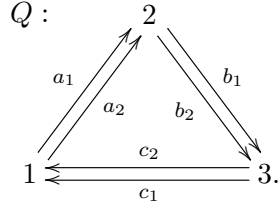
- a) the image $S_j(t)$ of the simple S_j under $\Phi(t)$ lies in \mathcal{A} or $\Sigma^{-1}\mathcal{A}$,
- b) the \mathbf{c} -vector $C(t)e_j$ equals $\underline{\dim} S_j(t)$.

The object $S_j(t)$ has endomorphism algebra k and does not have self-extensions since the object S_j has these properties in $\mathcal{D}(\Gamma(t))$. Thus, we find: each positive \mathbf{c} -vector is the dimension vector of a rigid indecomposable module over $J(Q, W)$. \square

4. EXAMPLES

We present some examples beyond the scope of our main result. An interesting question is to ask whether the inclusion in Theorem 14 is an equality. The following example shows that this is not the case in general.

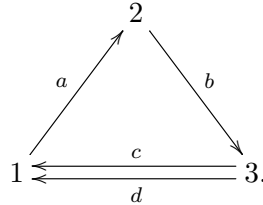
Example 15. Let Q be the Markov quiver



The potential $W = c_1 b_1 a_1 + c_2 b_2 a_2 - c_1 b_2 a_1 c_2 b_1 a_2$ makes $J(Q, W)$ finite dimensional as shown in [20, Example 8.2]. We know that the dimension vectors of the indecomposable projective modules are $(4, 4, 4)$ (see [6, Example 8.6]). By the description in [22] of the \mathbf{c} -vectors of this quiver, we know that $(4, 4, 4)$ is not a \mathbf{c} -vector.

Another possible direction for generalizing Theorem 11 is letting Q be mutation equivalent to an acyclic quiver. In an upcoming paper [23], it will be proved that if Q is mutation equivalent to a Dynkin quiver then the positive \mathbf{c} -vectors associated to Q are precisely the dimension vectors of indecomposable rigid modules over $J(Q, W)$ for a generic potential W . This is no longer true in more general settings, as the following example shows.

Example 16. Let Q be the quiver



This quiver is of type \tilde{A}_2 and $W = cba$ is a generic potential. The \mathbf{c} -vectors associated to Q are described in [4]. We can verify that $(1, 2, 1)$ is not a \mathbf{c} -vector, however it is the dimension vector of the indecomposable projective P_2 over $\text{mod}(J(Q, W))$.

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